Béziau's Contributions to the Logical Geometry of Modalities and Quantifiers

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Abstract. The aim of this paper is to discuss and extend some of Béziau's (published and unpublished) results on the logical geometry of the modal logic S5 and the subjective quantifiers *many* and *few*. After reviewing some of the basic notions of logical geometry, we discuss Béziau's work on visualising the Aristotelian relations in S5 by means of two- and three-dimensional diagrams, such as hexagons and a stellar rhombic dodecahedron. We then argue that Béziau's analysis is incomplete, and show that it can be completed by considering another three-dimensional Aristotelian diagram, viz. a rhombic dodecahedron. Next, we discuss Béziau's proposal to transpose his results on the logical geometry of the modal logic S5 to that of the subjective quantifiers *many* and *few*. Finally, we propose an alternative analysis of *many* and *few*, and compare it with that of Béziau's. While the two analyses seem to fare equally well from a strictly *logical* perspective, we argue that the new analysis is more in line with certain *linguistic* desiderata.

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1. Introduction

In recent years, Jean-Yves Béziau has been the driving force behind the renewed interest in the square of oppositions and related Aristotelian diagrams. On a practical level, he is the main organiser of a number of highly successful conference series and the editor-in-chief of a journal and a book series, all of which have functioned as a platform for discussion of recent discoveries about Aristotelian diagrams. On a theoretical level, Béziau has made significant contributions to the study of Aristotelian diagrams for a number of logical systems, such as the modal logic $\mathsf{S5}$ and the subjective quantifiers *many* and *few*. The main aims of this paper are to provide a detailed presentation of some of these (published and unpublished) results, to evaluate them from a logico-linguistic perspective, and finally, to show how they relate to the framework of logical geometry that we have recently been developing.

The paper is organized as follows. Section 2 provides a brief overview of some of the basic notions of logical geometry. Next, in Section 3 we discuss Béziau's work on visualising the Aristotelian relations in $\mathsf{S5}$ by means of two- and three-dimensional diagrams, such as hexagons and a stellar rhombic dodecahedron. We then argue in Section 4 that Béziau's analysis is incomplete, and show that it can be completed by considering another three-dimensional Aristotelian diagram, viz. a rhombic dodecahedron. In Section 5, we discuss Béziau's proposal to transpose his results on the logical geometry of the modal logic $\mathsf{S5}$ to that of the subjective quantifiers *many* and *few*. Next, in Section 6, we propose an alternative analysis of *many* and *few*. While the two analyses seem to fare equally well from a strictly *logical* perspective, we argue that the new analysis is more in line with certain *linguistic* desiderata. Section 7 provides a comparison between both analyses from the perspective of logical geometry, i.e. in terms of the various Aristotelian diagrams that they give rise to. Section 8, finally, wraps things up and mentions some questions for further research.

2. The basics of logical geometry

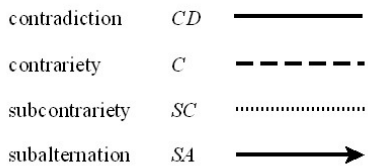
The logical geometry of a certain logical system or lexical field consists in the visual representation of the logical behaviour of its members. This behaviour can be classified according to a number of families of logical relations, such as the family of Aristotelian relations, the family of duality relations, etc. In the present paper, we will focus exclusively on the Aristotelian relations, i.e. contradiction (CD), contrariety (C), subcontrariety (SC) and subalternation (SA). Intuitively, the first three relations are defined in terms of whether the formulas can be true together (the $\varphi \wedge \psi$ part in the formal definition below) and whether they can be false together (the $\neg\varphi \wedge \neg\psi$ part in the formal definition below);[1] the fourth relation, SA, is defined in terms of truth propagation. Formally, the Aristotelian relations are defined relative to a logical system S:[2] two formulas φ and ψ are said to be

S-*contradictory*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
S-*contrary*	iff	$\mathsf{S} \models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \not\models \neg(\neg\varphi \wedge \neg\psi)$,
S-*subcontrary*	iff	$\mathsf{S} \not\models \neg(\varphi \wedge \psi)$	and	$\mathsf{S} \models \neg(\neg\varphi \wedge \neg\psi)$,
in S-*subalternation*	iff	$\mathsf{S} \models \varphi \rightarrow \psi$	and	$\mathsf{S} \not\models \psi \rightarrow \varphi$.

[1] It is well-known that $\neg(\neg\varphi \wedge \neg\psi)$ is equivalent to $\varphi \vee \psi$, but we choose to stick with the former notation, because it more clearly expresses the idea of φ and ψ being false together.

[2] The system S is assumed to have connectives expressing classical negation ($\neg$), conjunction ($\wedge$) and implication ($\rightarrow$), and a model-theoretic semantics ($\models$).

FIGURE 1. Code for visually representing the Aristotelian relations



These relations are abbreviated and visualized according to the code in Figure 1. When the system S is clear from the context, we will often leave it implicit, and simply talk about 'contrary' instead of 'S-contrary', etc.

In logical geometry we often use *bitstrings* as a compact way to represent the denotations of formulas [25, 28]. Bitstrings are, quite simply, sequences of bits (1 or 0), such as 101, 1100, 10110, etc. In the present paper we will exclusively work with bitstrings of length 4. Furthermore, we distinguish between bitstrings of *level 1* (L1), *level 2* (L2), and *level 3* (L3), which are defined as having a value 1 in one, two or three of their bit positions, respectively.[3] The Boolean operations on bitstrings are defined bitwise, for example $1100 \wedge 1010 = 1000$, $1100 \vee 1010 = 1110$ and $\neg 1100 = 0011$. Given the availability of these Boolean operations, we can reformulate the definitions of the Aristotelian relations in terms of bitstrings: two bitstrings φ and ψ are said to be

$$\begin{array}{llll}
\textit{contradictory} & \text{iff} \quad \varphi \wedge \psi = 0000 & \text{and} & \varphi \vee \psi = 1111, \\
\textit{contrary} & \text{iff} \quad \varphi \wedge \psi = 0000 & \text{and} & \varphi \vee \psi \neq 1111, \\
\textit{subcontrary} & \text{iff} \quad \varphi \wedge \psi \neq 0000 & \text{and} & \varphi \vee \psi = 1111, \\
\textit{in subalternation} & \text{iff} \quad \varphi \wedge \psi = \varphi & \text{and} & \varphi \vee \psi \neq \psi.
\end{array}$$

The squares in Figure 2 show the Aristotelian relations between three sets of formulas/bitstrings. Square (a) is the oldest and most widely known square, which is decorated with the quantified formulas of Aristotelian syllogistics, whereas square (b) represents the Aristotelian relations between four formulas from the modal logic $\mathsf{S5}$. Finally, square (c) represents the Aristotelian relations between four bitstrings 'in abstracto', i.e. without referring to any particular logical system or lexical field. Note that square (b) for the modal logic $\mathsf{S5}$ can be seen as a specific instance of square (c), if we take the bitstrings in the latter to be the denotations of the modal formulas in the former. Similar remarks apply to square (a).

The squares in Figure 2 have been generalized to larger and more complex Aristotelian diagrams. One classical example is the hexagon first studied by Jacoby, Sesmat and Blanché [7, 8, 15, 23]. Note that the square is not closed under the Boolean operators; for example, the square for $\mathsf{S5}$ contains the formulas $\Box p$ and $\neg\Diamond p$, but it does not contain their disjunction $\Box p \vee \neg\Diamond p$ (or any formula that is

[3]As limiting cases, the non-contingent bitstrings 0000 and 1111 can be called *level 0* (L0) and *level 4* (L4), respectively.

FIGURE 2. Aristotelian squares for (a) syllogistics, (b) the modal logic S5, and (c) bitstrings.

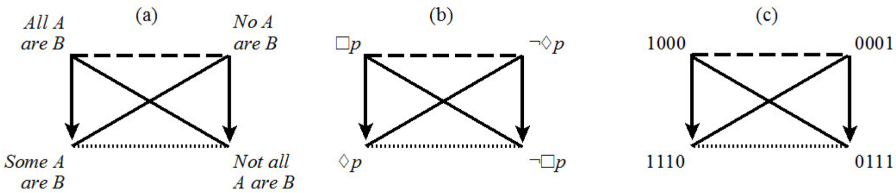
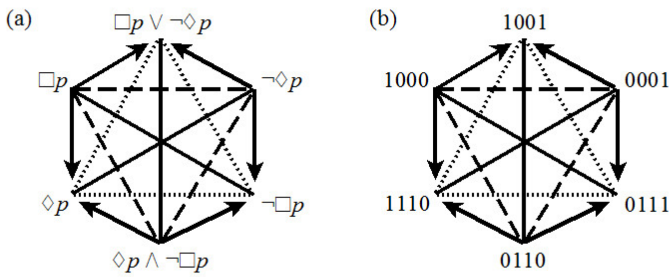


FIGURE 3. Aristotelian JSB hexagons for (a) the modal logic S5 and (b) bitstrings.



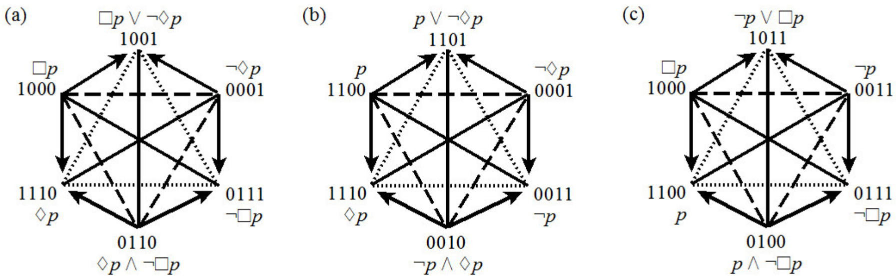
logically equivalent to it). By adding two extra vertices to the square, we obtain a hexagon that *is* Boolean closed.[4] Furthermore, the newly added formulas stand in Aristotelian relations to all formulas that were already present in the square and to each other, e.g. $\Box p \vee \neg\Diamond p$ is subcontrary to $\Diamond p$ and to $\neg\Box p$, and contradictory to $\Diamond p \wedge \neg\Box p$. The resulting configuration is an Aristotelian diagram called the *Jacoby-Sesmat-Blanché hexagon* (JSB); see Figure 3a. The abstract representation of this hexagon is the bitstring JSB hexagon in Figure 3b. Equivalently, the latter can be seen as the Boolean closure of the square in Figure 2c. For example, the bitstring representation of the disjunction $\Box p \vee \neg\Diamond p$ is 1001, which is the join of the bitstrings 1000 and 0001, representing the formulas $\Box p$ and $\neg\Diamond p$, respectively.

3. Béziau on the logical geometry of S5

We now turn to Béziau's results on the logical geometry of S5 [1, 2, 5]. Starting from the Aristotelian square in Figure 2b, he constructs three JSB hexagons for S5.

[4]Formally, a diagram or set of formulas is said to be *Boolean closed* iff whenever it contains formulas φ, ψ, it also contains their contingent Boolean combinations ($\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$). Note that this definition is restricted to *contingent* Boolean combinations, so if $\Box p$ and $\neg\Box p$ occur in a diagram, then it is *not* required for the diagram to be Boolean closed that it also contains the contradiction $\Box p \wedge \neg\Box p$ and the tautology $\Box p \vee \neg\Box p$.

FIGURE 4. Béziau's three JSB hexagons for S5: (a) the classical, (b) the paracomplete, and (c) the paraconsistent hexagon.



The first hexagon is simply the Boolean closure of the original square, which was already discussed above (see Figure 3a), and which is repeated here as Figure 4a. The two other hexagons are the Boolean closures of two new squares, which are obtained from the original square by replacing one of its diagonals with p—$\neg p$. The first new square is obtained by replacing the $\Box p$—$\neg\Box p$ diagonal, and gives rise to the JSB hexagon in Figure 4b. Similarly, the second new square is obtained by replacing the $\Diamond p$—$\neg\Diamond p$ diagonal, and gives rise to the JSB hexagon in Figure 4c.[5]

Béziau used these JSB hexagons to show that the modal logic S5 can model classical as well as non-classical modes of reasoning. The hexagon in Figure 4b shows the relation between classical negation ($\neg$) and paracomplete negation ($\neg\Diamond$). Classical negation is a contradictory-forming operator: p and $\neg p$ are contradictory, and thus cannot be true together nor false together. By contrast, paracomplete negation is 'merely' a contrary-forming operator: p and $\neg\Diamond p$ are contrary, and thus can be false together. Similarly, the hexagon in Figure 4c shows the relation between classical negation ($\neg$) and paraconsistent negation ($\neg\Box$). Paraconsistent negation is 'merely' a subcontrary-forming operator: p and $\neg\Box p$ are subcontrary, and thus can be true together.[6]

The three JSB hexagons in Figure 4 contain a total number of 12 formulas.[7] Béziau suggested that by moving from two-dimensional to three-dimensional

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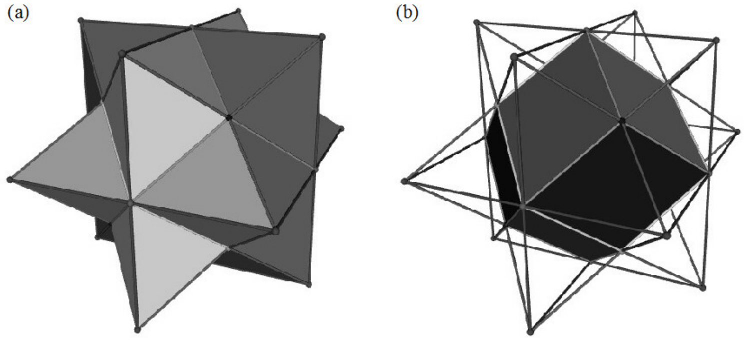
[5]The process described above works not only for the S5 square (Figure 2b), but also for the bitstring square (Figure 2c). We thus obtain three bitstring JSB hexagons, the first one of which was already shown in Figure 3b. The two new ones are displayed simultaneously with their S5 counterparts in Figure 4. For reasons of space, we will in the remainder of this paper no longer distinguish between S5 diagrams and bitstring diagrams, and decorate the Aristotelian diagrams sometimes with concrete formulas, sometimes with bitstrings, and sometimes with both simultaneously.

[6]These hexagons also show that, from the perspective of subalternation, classical negation occupies a position that is intermediate between paracomplete and paraconsistent negation. In Figure 4b, the classical negation ($\neg p$) is *entailed by* the paracomplete negation ($\neg\Diamond p$), while in Figure 4c, it *entails* the paraconsistent negation ($\neg\Box p$).

[7]Since there are 3 hexagons and each hexagon contains 6 formulas, one might expect the total number to be $3 \times 6 = 18$ formulas. However, this calculation ignores the fact that certain formulas

Figure 5. (a) Béziau's stellar rhombic dodecahedron, (b) the stellar rhombic dodecahedron as the first stellation of the rhombic dodecahedron.



diagrams, these 12 formulas can be visualised quite elegantly by means of a single Aristotelian diagram, viz. a *stellar rhombic dodecahedron* (Figure 5a). This polyhedron is also known as 'Escher's solid'; geometrically speaking, it is the first stellation of the rhombic dodecahedron [17] (Figure 5b).[8] Furthermore, Béziau remarked that if we ignore the subalternations on the outside edges of the three JSB hexagons in Figure 4, we obtain three JSB *stars* (each star consists of a triangle of contraries interlocked with a triangle of subcontraries). Each of these three JSB stars can be embedded inside Béziau's stellar rhombic dodecahedron; see Figure 6.

Soon after Béziau's discoveries, H. Smessaert and A. Moretti (then a PhD student of Béziau's) independently observed that a fourth JSB star can be constructed using the 12 formulas appearing in Béziau's stellar rhombic dodecahedron. Just as before, this JSB star can be seen as a JSB hexagon with the subalternations left out; see Figure 7a. This fourth JSB star can also be embedded inside the stellar rhombic dodecahedron (Figure 7b), and therefore each of its six formulas already appears in one of the first three JSB stars/hexagons in Figure 4. In other words, the novelty of the fourth JSB star does not consist in its formulas

occur in two distinct hexagons. In particular, the formulas $\Diamond p$ and $\neg\Diamond p$ occur in hexagons (a) and (b), the formulas $\Box p$ and $\neg\Box p$ occur in hexagons (a) and (c), and the formulas p and $\neg p$ occur in hexagons (b) and (c).

[8]Somewhat confusingly, Béziau [1] talks about the 'stellar dodecahedron' instead of the 'stellar *rhombic* dodecahedron', thereby suggesting that the solid he had in mind (but never actually drew) is the stellation of the 'ordinary' *pentagonal* (i.e. Platonic) dodecahedron, rather than that of a *rhombic* dodecahedron. This confusion resurfaces in Moretti's remarks that Béziau's solid is "obtained by constructing a *pentagonal* pyramid or spike over each of the 12 *pentagonal* faces of a dodecahedron" [19, p. 75, our emphases]. Accordingly, the figure given by Moretti [19, p. 76] shows the stellation of a pentagonal dodecahedron, rather than that of a rhombic dodecahedron (although he still calls it 'Escher's solid' and attributes it to Béziau). In a more recent paper, Béziau does provide a figure of the stellar rhombic dodecahedron (without its S5-decoration) [5, p. 13], but still calls it the 'stellar dodecahedron'.

FIGURE 6. Béziau's three JSB stars embedded inside the stellar rhombic dodecahedron.

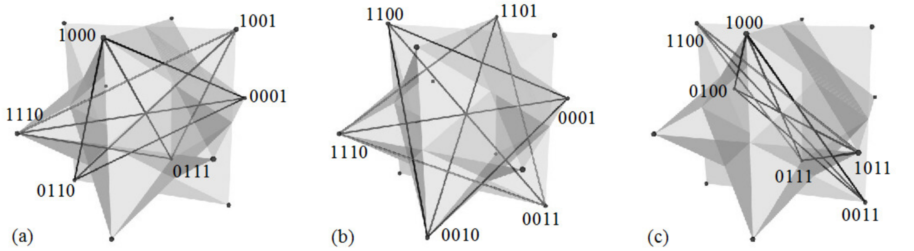
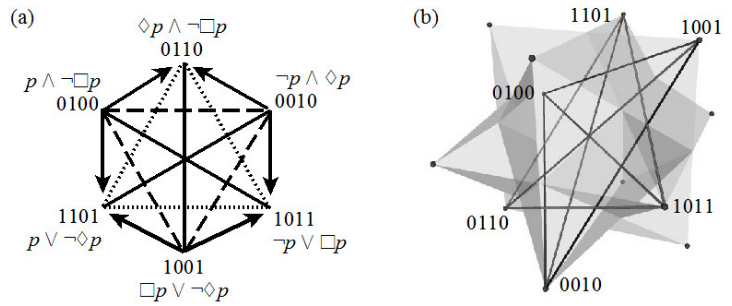


FIGURE 7. (a) The fourth JSB hexagon; (b) the corresponding JSB star embedded inside the stellar rhombic dodecahedron.

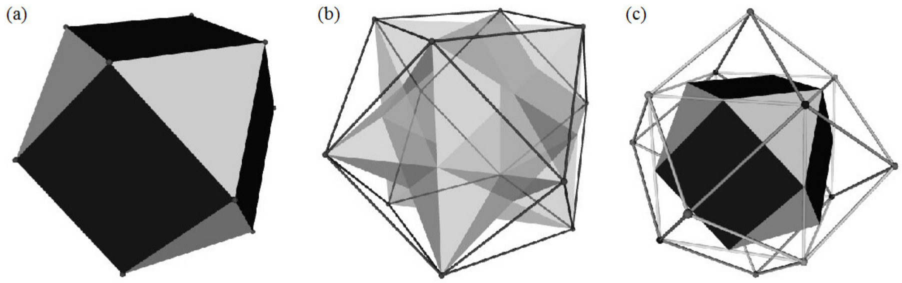


(these were already present in the previous stars), but rather in the fact that the pattern of Aristotelian relations between these formulas is again that of a JSB star/hexagon. Furthermore, by taking into account this fourth JSB star, we see that the stellar rhombic dodecahedron achieves a certain 'equilibrium':[9] on the one hand, each of the 12 formulas of the stellar rhombic dodecahedron appears in 2 JSB stars ($12 \times 2 = 24$); on the other hand, there are 4 JSB stars embedded inside the stellar rhombic dodecahedron, each of which contains 6 formulas ($4 \times 6 = 24$).

Dissatisfied with Béziau's decision to ignore the subalternation relations (i.e. to move from the JSB *hexagons* to the corresponding JSB *stars*), Moretti [18] went on to look for alternative polyhedra to represent the 12 formulas and their Aristotelian relations (including the subalternations). He proposed the *cuboctahedron* (Figure 8a), and showed that the four JSB hexagons can be embedded inside it (Figure 9)—just like the four corresponding JSB stars can be embedded inside Béziau's stellar rhombic dodecahedron (recall Figures 6 and 7b). Interestingly, Moretti's cuboctahedron turns out to be the convex hull of Béziau's stellar rhombic dodecahedron (Figure 8b)—just like the JSB hexagons are the convex hulls of

[9]Also recall Footnote 7.

FIGURE 8. (a) Moretti's cuboctahedron, (b) the cuboctahedron as the convex hull of Béziau's stellar rhombic dodecahedron, (c) the cuboctahedron as the dual polyhedron of the rhombic dodecahedron.



the corresponding stars. This geometric observation has a direct logical analogue: Moretti's cuboctahedron can be seen as the result of adding the subalternation relations to Béziau's stellar rhombic dodecahedron, and the four JSB hexagons are embedded in the former in exactly the same way as the corresponding JSB stars are embedded in the latter (compare Figures 6 and 7b with Figure 9). Finally, it should be noted that Moretti's cuboctahedron is the dual polyhedron of a rhombic dodecahedron (Figure 8c).

4. Extending Béziau's results on the logical geometry of $\mathsf{S5}$

We have seen above that the 12 formulas of $\mathsf{S5}$ that were considered by Béziau give rise to four distinct JSB hexagons. Each of these hexagons is, by itself, closed under the Boolean operators. However, when the 12 formulas are taken together (as is done in Béziau's stellar rhombic dodecahedron and Moretti's cuboctahedron), the resulting diagram is *not* Boolean closed. For example, it contains the formulas[10] $\Box p$ and $\neg p \wedge \Diamond p$, but it does not contain their disjunction $\Box p \vee (\neg p \wedge \Diamond p)$ (or any formula that is logically equivalent to it). Reformulating the example in terms of bitstrings: Béziau's analysis deals with the bitstrings 1000 and 0010, but not with their join 1010. This shows that Béziau's analysis is incomplete.

In order to obtain the Boolean closure of Béziau's analysis, we have to consider two additional formulas, viz. $\Box p \vee (\neg p \wedge \Diamond p)$ (which was already mentioned above) and its negation, $\neg\Box p \wedge (p \vee \neg\Diamond p)$. Note that these two formulas are syntactically more complex than any of the 12 formulas considered by Béziau, which might explain why these are exactly the two that were not included in his original analysis. In terms of bitstrings, the two new formulas correspond to the two new

[10]Note, trivially perhaps, that these two formulas do not occur together in any of the four JSB hexagons considered above. After all, if a diagram contains these two formulas, then it cannot be Boolean closed.

FIGURE 9. Four JSB hexagons embedded inside Moretti's cuboctahedron.

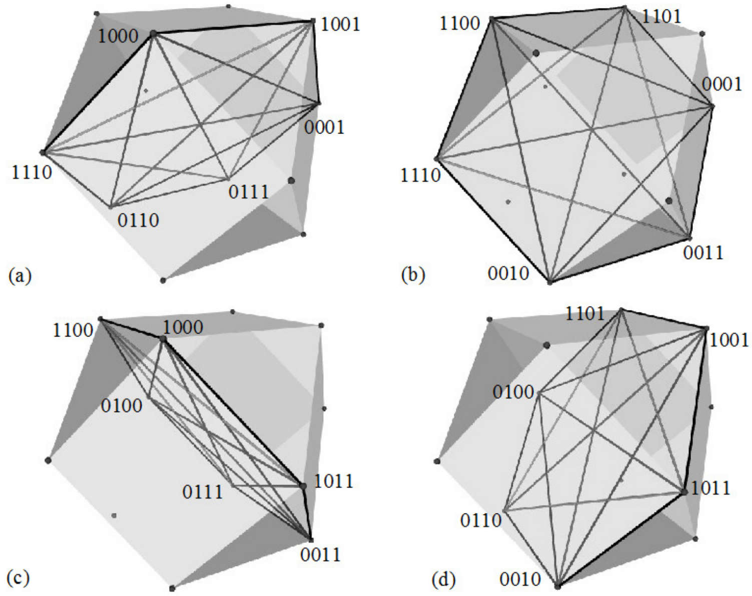


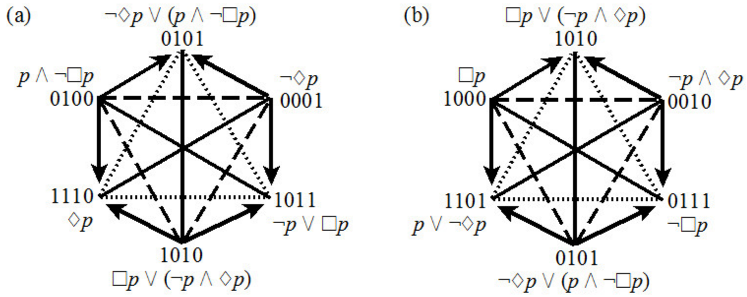
TABLE 1. The 14 formulas of S5 and their bitstring representations.

S5-formula	bitstring	S5-formula	bitstring
$\Box p$	1000	$\neg\Box p$	0111
$\neg\Diamond p$	0001	$\Diamond p$	1110
$\Box p \vee \neg\Diamond p$	1001	$\neg\Box p \wedge \Diamond p$	0110
p	1100	$\neg p$	0011
$p \wedge \neg\Box p$	0100	$\neg p \vee \Box p$	1011
$\neg p \wedge \Diamond p$	0010	$p \vee \neg\Diamond p$	1101
$\Box p \vee (\neg p \wedge \Diamond p)$	1010	$\neg\Box p \wedge (p \vee \neg\Diamond p)$	0101

bitstrings 1010 and 0101, respectively. It is easy to see that by adding these 2 formulas/bitstrings to the 12 considered by Béziau, we obtain a set of 14 formulas/bitstrings that *is* Boolean closed; see Table 1.

Of course, the two new formulas enter into a variety of Aristotelian relations with the 12 old ones. In particular, Smessaert [25] noted that they yield two new JSB hexagons, which are shown in Figure 10. Furthermore, it should be stressed that because of combinatorial reasons, the list of six JSB hexagons that we have now obtained (comprising the four that were already present in Béziau's analysis,

Figure 10. Two new JSB hexagons for S5.



together with the two new ones) is *exhaustive*, i.e. there are no additional JSB hexagons that can be constructed with the 14 S5-formulas under consideration.[11]

The question now arises as to how this set of 14 formulas (and the Aristotelian relations between them) can be visualised by means of a three-dimensional Aristotelian diagram. There have recently been a number of related—albeit subtly different—proposals. Smessaert [25, 26] and Demey [10] make use of a *rhombic dodecahedron*. Moretti [18] and Pellissier [20] make use of a so-called *tetraicosahedron*. Finally, it has recently been discovered that this visualisation issue was already discussed in full detail in the 1960s by P. Sauriol, who made use of a so-called *tetrahexahedron* [22].[12] Of these three, only the rhombic dodecahedron is canonically discussed in the mathematical literature on polyhedra [9] and does equal justice to its cube and octahedron components [27]; see Figure 11. Furthermore, as was already noted above, the rhombic dodecahedron is geometrically related to both of the three-dimensional Aristotelian diagrams that were discussed in the previous section: Béziau's stellar rhombic dodecahedron is its first *stellation* (recall Figure 5b), while Moretti's cuboctahedron is its *dual* polyhedron (recall Figure 8c). Finally, the rhombic dodecahedron fits naturally in a unified perspective on Aristotelian diagrams and Hasse diagrams [12]. Because of these reasons, we will henceforth use the rhombic dodecahedron as the canonical representation of the logical geometry of S5.

Because its set of 14 formulas is Boolean closed, the rhombic dodecahedron constitutes a natural endpoint in the analysis of the logical geometry of S5. Putting it in terms of bitstrings, the rhombic dodecahedron provides a complete account of the logical geometry of bitstrings *of length 4*: every Aristotelian diagram that can be constructed with bitstrings of length 4, can be embedded inside the rhombic

[11] Pellissier [20] and Moretti [18] distinguish between a strong and a weak kind of JSB hexagon (the strong kind is the kind we have considered up till now), and note that the 14 formulas not only yield 6 strong JSB hexagons, but also 4 weak JSB hexagons.
[12] The differences between these three visualisations are discussed in more detail in [27, Section 2].

FIGURE 11. (a) The rhombic dodecahedron, (b) the rhombic dodecahedron as the compound of a cube and an octahedron.

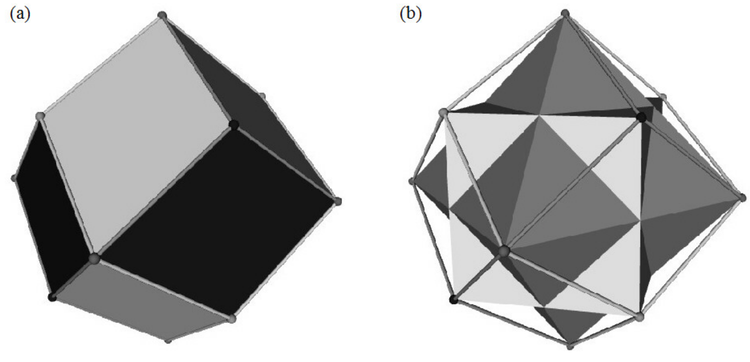
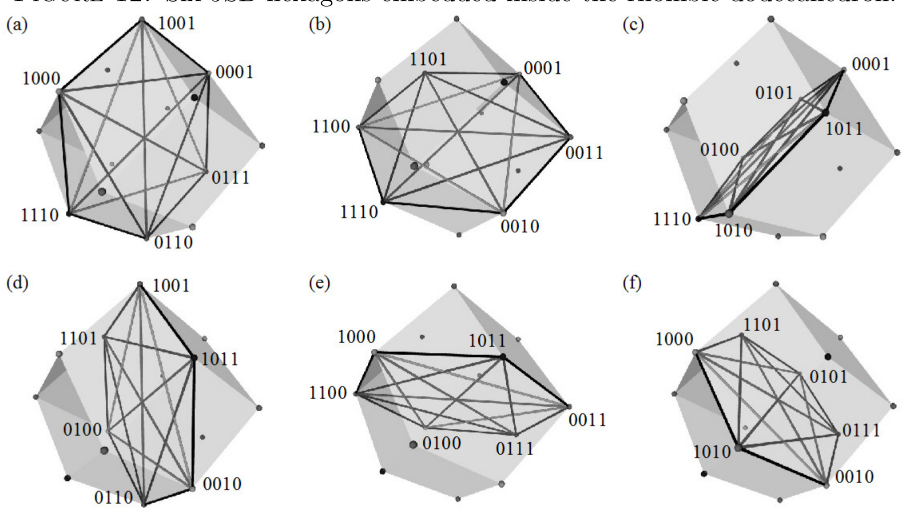


FIGURE 12. Six JSB hexagons embedded inside the rhombic dodecahedron.



dodecahedron.[13] We are currently developing a systematic typology of all these diagrams [30]; however, for our current purposes, it suffices to note that each of the six JSB hexagons (recall Figures 4, 7b and 10) can be embedded inside the rhombic dodecahedron; see Figure 12.

[13]Going beyond the rhombic dodecahedron would thus require us to introduce bitstrings of length 5. This can certainly be done, but the Aristotelian diagrams will become exponentially larger. For example, as far as Boolean closed diagrams are concerned, we move from the rhombic dodecahedron (which has $2^4 - 2 = 14$ vertices) to a diagram that has $2^5 - 2 = 30$ vertices. (Recall Footnotes 3 and 4.)

TABLE 2. The analogy between S5 and the (subjective) quantifiers.

S5-formula	bitstring	(subjective) quantifiers
$\Box p$	1000	*all*
$\neg\Box p$	0111	*not all*
$\neg\Diamond p$	0001	*no*
$\Diamond p$	1110	*at least one*
$\Box p \vee \neg\Diamond p$	1001	*no or all*
$\neg\Box p \wedge \Diamond p$	0110	*some*
p	1100	$many_1$
$\neg p$	0011	few_1
$p \wedge \neg\Box p$	0100	$many_1$ *but not all*
$\neg p \wedge \Diamond p$	0010	*at least one but* few_1
$\neg p \vee \Box p$	1011	*all or* few_1
$p \vee \neg\Diamond p$	1101	*no or* $many_1$
$\Box p \vee (\neg p \wedge \Diamond p)$	1010	*all or (at least one but* few_1*)*
$\neg\Box p \wedge (p \vee \neg\Diamond p)$	0101	*no or (*$many_1$ *but not all)*

5. Béziau on the logical geometry of the subjective quantifiers

Over the years, Béziau has transposed his analysis of the logical geometry of S5 to a number of other fields. For example, he has recently shown how one can also construct JSB hexagons for metalogical notions such as theoremhood and consistency [6]. In the second part of this paper, we will focus on another application of his analysis, viz. the subjective quantifiers, which he presented in 2008 at the LNAT 1 conference in Brussels [3], but which has remained unpublished so far.

Béziau's starting point is the observation that the logical geometry of S5 seems analogous to that of a certain set of quantifiers, including the subjective quantifiers *many* and *few*. In more abstract terms, both the formulas from S5 and the lexical field of subjective quantification can be given a semantics in terms of bitstrings of length 4. The exact analogy proposed by Béziau is described in Table 2. The first six rows comprise the standard universal and existential quantifiers and their Boolean combinations; the next eight rows comprise the subjective quantifiers $many_1$ and few_1 and their Boolean combinations with each other and with the first six.[14]

The analogy between S5-formulas and quantifiers in the first six rows is very natural, given the familiar Kripke semantics of modal logic in terms of quantification over possible worlds. For example, the truth of $\Box p$ consists in p being true in *all* (accessible) possible worlds, while the truth of $\Diamond p$ consists in p being true in *at least one* (accessible) possible world. Note that Béziau explicitly distinguishes between the two existential expressions *at least one* and *some*: for example, 'at least

[14] We will henceforth add a '1' in subscript to the expressions *many* and *few* to refer to the semantic interpretation they receive in Béziau's analysis. Similarly, in the next sections, we will add a '2' in subscript to refer to our alternative analysis.

one A is B' does not exclude the possibility that *all* As are Bs, but 'some As are B' does exclude this possibility (i.e. it entails that at least one A is not B). This distinction corresponds to the linguistic distinction between the 'one-sided' and 'two-sided' readings of the existential quantifier [13].[15,16] In terms of the Boolean operators, we have the following equivalences:

$$
\begin{array}{rcll}
\textit{some} & \equiv & \textit{at least one but not all} & (0110 = 1110 \wedge 0111) \\
\textit{at least one} & \equiv & \textit{some or all} & (1110 = 0110 \vee 1000)
\end{array}
$$

There is disagreement among linguists whether the two-sided reading of the natural language expression *some* is a matter of semantics or pragmatics [13, 24]; Béziau thus sides with those who take it to be a matter of semantics. Finally, it should be noted that these six quantifiers yield an alternative decoration for the JSB hexagon shown in Figure 4a.

As far as the bottom eight rows of Table 2 are concerned, the core of Béziau's analysis consists in treating the subjective quantifiers $many_1$ and few_1 on a par with the so-called 'null-modalities' in $\mathsf{S5}$, namely the formulas p and $\neg p$, which do not contain a modal operator. In $\mathsf{S5}$, we start with a tripartition of logical space into 'necessity' (1000), 'contingency' (0110) and 'impossibility' (0001), and superimpose upon it a bipartition into 'actually true' (1100) and 'actually false' (0011). Keeping in mind the Kripke semantics of modal logic described above, the space of quantification can be tripartitioned by means of the expressions *all* (1000), *some* (0110) and *no* (0001). Béziau's analysis now superimposes a bipartition by means of the subjective quantifier expressions $many_1$ (1100) and few_1 (0011). The entailments in $\mathsf{S5}$ from the level 1 (L1) notion of 'necessity' (1000) to the level 2 (L2) notion of 'actual truth' (1100) and from the L1 notion of 'impossibility' (0001) to the L2 notion of 'actual falsehood' (0011) get straightforward counterparts in the realm of subjective quantification. More in particular, $many_1$ and few_1 are L2 elements: $many_1$ (1100) is entailed by *all* (1000), whereas few_1 (0011) is entailed by *no* (0001). This accounts for the first two rows of the bottom part of Table 2, which constitute the core of Béziau's analogy between the modalities and the quantifiers.

The remaining six rows are then built by means of the Boolean operators of conjunction and disjunction. The purpose of the first pair of Boolean combinations, i.e. the conjunctions $many_1$ *but not all* and *at least one but* few_1, is to create the L1 elements 0100 and 0010 by excluding the extreme values of the tripartition, namely *all* (1000) and *no* (0001), respectively. The two disjunctions *all or* few_1 and *no or* $many_1$ yield the L3 elements 1011 and 1101. The final two quantifier

[15] In terms of bitstrings, the one-sided reading corresponds to a bitstring that has *one* transition in bit values, i.e. from 1 to 0 or vice versa (e.g. 1110), whereas the two-sided reading corresponds to a bitstring having *two* transitions in bit values (e.g. 0110).

[16] This distinction also applies to the modal operators, where one-sided possibility ($\Diamond p$) is compatible with necessity, but two-sided possibility ($\Diamond p \wedge \neg\Box p$, usually called 'contingency') is not.

expressions,[17] *all or (at least one but few$_1$)* and *no or (many$_1$ but not all)*, have a layered Boolean structure (in that the top level disjunction has a second disjunct which is itself a conjunction) and correspond to the L2 elements 1010 and 0101, respectively. Recall from Section 4 that also in $\mathsf{S5}$, it is precisely these two L2 elements which correspond to the most complex formulas. More in general, the bottom part of Table 2 reveals a very strong parallellism between the formulas of $\mathsf{S5}$ and the subjective quantifiers in terms of lexico-syntactic complexity, i.e. the minimal[18] number of binary connectives they require: the two basic elements (1100 and 0011) contain *no* binary connective, the next four elements (0100, 0010, 1011 and 1101) get *one* binary connective, and the final two (1010 and 0101) have *two* binary connectives.

6. An alternative analysis of the subjective quantifiers

In this section we propose an alternative analysis for the logical geometry of the subjective quantifiers based on linguistic considerations.[19] Notice, by the way, that Béziau's own distinction between the two-sided *some* and the one-sided *at least one* provides the perfect starting point for such an alternative analysis. From the perspective of lexicalisation, i.e. the amount of lexical material an expression consists of, *some* is more primitive than *at least one*. The semantic complexity in terms of the levels of the corresponding bitstrings then runs perfectly parallel to this lexical complexity: the L2 bitstring 0110 for *some* is less complex than the L3 bitstring 1110 for *at least one*.[20]

This correlation between lexical and semantic complexity no longer holds, however, with the subjective quantifier expressions in the bottom part of Table 2. First of all, *many$_1$* is lexically more primitive than *many$_1$ but not all*, but the former's L2 bitstring (1100) is semantically more complex than the latter's L1 bitstring (0100). Completely analogously, *few$_1$* is lexically more primitive than *at least one but few$_1$*, but the former's L2 bitstring (0011) is again semantically more complex than the latter's L1 bitstring (0010). Similar discrepancies can be observed the next level up: *no or many$_1$* is lexically less complex than *no or (many$_1$*

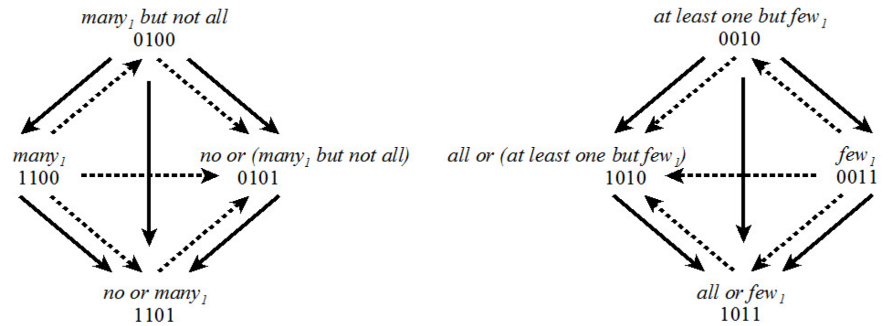
[17]Strictly speaking, these two quantifier expressions did not feature in Béziau's presentation [3], but as was argued in Section 4, they can straightforwardly be added by taking the Boolean closure of the original set of 12 expressions.

[18]We only look at the *minimal* number of binary connectives, because every semantic value (bitstring) can be expressed in a number of syntactically different ways. For example, the bitstring 0010 can be expressed as $\neg p \wedge \Diamond p$ [0011 $\wedge$ 1110] with *one* binary connective, but also as $\neg p \wedge \big(\Box p \vee (\neg p \wedge \Diamond p)\big)$ [0011 $\wedge$ 1010] with *three* binary connectives.

[19]From a linguistic point of view, the semantics of *many* and *few* is notoriously complex. For example, Keenan writes: "we shall largely exclude *many* and *few* from the generalizations we propose since our judgments regarding their interpretations are variable and often unclear [16, p. 47–48].

[20]If we were to work with bitstrings of length 3, ignoring the subjective quantifiers and focussing on the six quantifier expressions in the top part of Table 2, *some* would be the L1 bitstring 010, whereas *at least one* would be the L2 bitstring 110.

FIGURE 13. Semantic and lexical complexity in Béziau's analysis.



but not all), but nevertheless the former is L3 (1101) whereas the latter is only L2 (0101). The mismatches between semantic and lexical complexity are visually represented in Figure 13. The full line arrows represent the semantic complexity increasing from L1 at the top to L3 at the bottom, thus reflecting the entailment or subalternation relation. The dashed line arrows, by contrast, reflect the increase in lexical complexity. In 8 out of the 12 cases, there is a mismatch between the increases in semantic and lexical complexity.[21] In more visual terms, the 'orientation' of the two lattices for semantic complexity in Figure 13 is *from the top downwards*, whereas that of the two lattices for lexical complexity is *from the outside inwards* (i.e. with $many_1$ and few_1 as their respective starting points).

In view of these considerations, the key property of our alternative analysis is that *many* and *few* are characterized as the L1 bitstrings 0100 and 0010, respectively. As a consequence, $many_2$ is incompatible with *all*: the bitstrings 0100 and 1000 are contrary. In contrast, recall that Béziau's $many_1$ (1100) is entailed by *all* (1000). This entailment is due to the analogy Béziau draws between modalities and quantifiers: just like $\Box p$ (1000) entails p (1100), he takes *all* (1000) to entail $many_1$ (1100). Although the former entailment is beyond any doubt, the latter is more questionable. Consider a situation, for instance, in which the universe of discourse contains three books, all three of which have been read by John. In this situation, the proposition *John has read all books* is obviously true, but the proposition *John has read many books* is very likely to be considered false, for the simple reason that 'three books' do not really count as 'many books'. In other words, *all* need not entail *many*, although on many occassions it will actually do so, of course. In order to reflect this possible absence of entailment, our alternative

[21] More specifically, three types of mismatches can be distinguished: (i) there is both a semantic and a lexical arrow but they point in opposite directions—e.g. between 1100 and 0100, (ii) there is a semantic arrow but no lexical arrow at all—e.g. between 0100 and 1101, and (iii) there is a lexical arrow but no semantic arrow at all—e.g. between 1100 and 0101.

analysis of the subjective quantifiers assigns a two-sided reading to natural language *many*:[22] whereas Béziau's $many_1$ is the one-sided L2 element 1100 (which has a single transition in bit values), our $many_2$ is the two-sided L1 bitstring 0100 (which has two transitions in bit values).[23] This analysis is further supported by the fact that a lexically complex expression such as *many if not all* exactly allows us to turn the two-sided $many_2$ into a one-sided reading, by incorporating the *all* in the disjunction,[24] thus retrieving the L2 semantics 1100 of Béziau's $many_1$:

$$many_1 \quad \equiv \quad many_2 \ or \ all / many_2 \ if \ not \ all \qquad (1100 = 0100 \vee 1000)$$

A largely analogous story can now be told for the negative subjective quantifier *few*. For Béziau, the validity of the modal entailment from $\neg\Diamond p$ (0001) to $\neg p$ (0011) carries over to that from *no* (0001) to few_1 (0011). Once again, however, the latter entailment is somewhat problematic. Concluding from the truth of *John has read no books* to that of *John has read few books* runs into conflict with the existential presupposition that seems to accompany the latter proposition: qualifying the amount of books read as 'few' requires that there exists 'at least one' book read. In order to do justice to this intuition, few_2 receives a two-sided L1 analysis which is incompatible with *no*: the respective bitstrings 0010 and 0001 are contrary. Here as well, the analysis is further supported by the existence of lexically complex expressions such as *few if any*. The two-sided few_2 changes into a one-sided reading, by incorporating the *no* in the disjunction,[25] thus recovering the L2 semantics 0011 of Béziau's few_1:

$$few_1 \quad \equiv \quad few_2 \ or \ no / few_2 \ if \ any \qquad (0011 = 0010 \vee 0001)$$

Whereas Béziau's $many_1$ (1100) and few_1 (0011) are contradictories and thus yield a partition of the entire logical space, our own $many_2$ (0100) and few_2 (0010) are merely contraries and yield a more fine-grained partition of the two-sided quantifier *some* (0110):

$$some \quad \equiv \quad many_2 \ or \ few_2 \qquad (0110 = 0100 \vee 0010)$$

[22]Notice, incidentally, that the difference in subscripts between Béziau's $many_1$ and our $many_2$ nicely reflects this contrast between the *one*-sided and the *two*-sided readings.
[23]Recall Footnote 15.
[24]Using simple propositional reasoning, the expression *many if not all* can be shown to be equivalent to the expression *many or all*. Intuitions differ as to whether an expression of the form *p if not q* should be read as the conditional $\neg p \to q$ or rather as $\neg q \to p$, but both readings are equivalent to the disjunction $p \vee q$.
[25]Since *any* can be seen as the negation of *no*, the expression *few if any* is semantically equivalent to *few if not no*, and can thus also be shown to boil down to the disjunction *few or no* (recall Footnote 24). Additional linguistic evidence for this equivalence comes from translational equivalents such as Dutch *weinig of geen* and French *peu ou pas*. Furthermore, even in English the disjunctive semantics of *few or no* is lexicalized, albeit only for abstract and mass nouns, viz. as *little or no*.

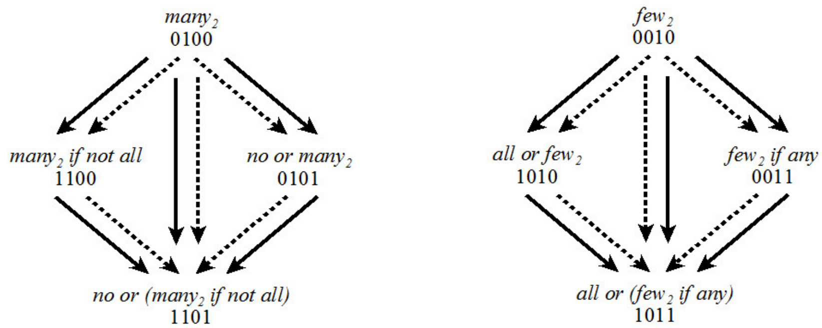
TABLE 3. An alternative to Béziau's analysis of the (subjective) quantifiers.

Béziau's analysis	bitstring	alternative analysis
all	1000	*all*
not all	0111	*not all*
no	0001	*no*
at least one	1110	*at least one*
no or all	1001	*no or all*
some	0110	*some*
$many_1$	1100	$many_2$ *if not all*
few_1	0011	few_2 *if any*
$many_1$ *but not all*	0100	$many_2$
at least one but few_1	0010	few_2
all or few_1	1011	*all or (*few_2 *if any)*
no or $many_1$	1101	*no or (*$many_2$ *if not all)*
all or (at least one but few_1*)*	1010	*all or* few_2
no or ($many_1$ *but not all)*	0101	*no or* $many_2$

Continuing along these lines, we can calculate all the Boolean combinations of our $many_2$ and few_2, and thereby obtain a Boolean closed set of 14 quantifier expressions (just like in Béziau's analysis). Table 3 presents a comparative overview of Béziau's and our analyses. Note that both analyses make use of the same set of 14 bitstrings, and will thus have exactly the same logical properties (e.g. the same types and numbers of Aristotelian relations); their differences are thus purely a matter of how these bitstrings are mapped onto the concrete natural language expressions. More specifically, we see that both analyses agree on this mapping for the first six expressions, i.e. on the 'ordinary' universal and existential quantifiers (and their Boolean combinations). The differences in the two mappings are thus entirely situated in the final eight expressions, i.e. those involving the subjective quantifiers *many* and *few*.

We have argued for our alternative analysis by appealing to logical intuitions (e.g. concerning the entailments between *all* and *many*) as well as lexicalisation patterns in natural languages (e.g. *little or no*, *weinig of geen*, *peu ou pas*, etc.). Additionally, our alternative analysis turns out to avoid the mismatches between semantic and lexical complexity in Béziau's analysis, which were shown in Figure 13. The corresponding diagram for our own analysis is shown in Figure 14: full line arrows still represent increases in semantic complexity (entailment/subalternation), while dashed line arrows represent increases in lexical complexity. Recall that in Béziau's analysis, there is a mismatch between semantic and lexical complexity

FIGURE 14. Semantic and lexical complexity in the alternative analysis.



in 8 out of the 12 cases. In the alternative analysis, however, there are no mismatches whatsoever: the lattices for semantic and lexical complexity share a single 'orientation', viz. *from the top downwards*.[26]

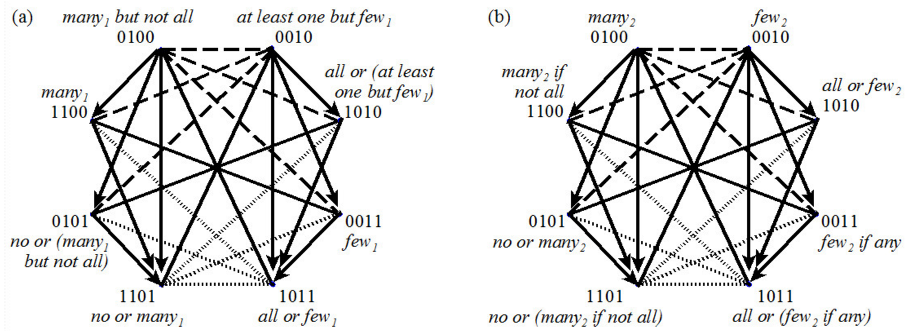
7. Aristotelian diagrams for the subjective quantifiers

In this section we compare our alternative analysis of the subjective quantifiers with that of Béziau's from the perspective of logical geometry, i.e. in terms of the various Aristotelian diagrams they give rise to. Considering Table 3 from Section 6, we already noted that both analyses entirely agree with each other on the first six bitstrings. Furthermore, in Section 5 these six bitstrings were shown to yield a JSB hexagon, which was displayed in Figure 4a.

The differences between both analyses are thus entirely situated within the next eight bitstrings of Table 3, i.e. the subjective quantifiers *many* and *few* (and their Boolean combinations). The diagrams in Figures 13 and 14 can be seen as partial Aristotelian diagrams for Béziau's and our analysis of the subjective quantifiers, respectively, in the sense that the full line arrows in these diagrams represent subalternation, which is one of the four Aristotelian relations. One might consider turning these diagrams into full-fledged Aristotelian diagrams by adding the other Aristotelian relations, but this leads to a suboptimal visualisation. For example, the bitstrings 1100 and 0011 are contradictory to each other, and similarly for 0101 and 1010, but since these four bitstrings are collinear—they lie on a single (horizontal) line—, the 'short' contradiction edge between 0101 and 1010 would

[26]Note that there is no horizontal semantic arrow between the bitstrings 1100 and 0101, since neither of them entails the other one. There is no horizontal lexical arrow between them either, since their corresponding quantifier expressions have the same degree of lexical complexity (viz. a single Boolean operator). Because of this twofold absence, the correlation between semantic and lexical complexity is preserved in this case as well. Similar remarks apply to the case of 1010 and 0011.

Figure 15. Buridan octagons for the subjective quantifiers: (a) Béziau's analysis, (b) our alternative analysis.



not be visually distinguishable from the 'long' contradiction edge between 1100 and 0011.

We therefore turn to an alternative visualisation, which has the shape of a convex octagon. This diagram was first used as an Aristotelian diagram by the medieval logician John Buridan, and is therefore canonically called the *Buridan octagon* [14, 21, 27, 29]. Figure 15 shows Buridan octagons for Béziau's analysis and our own analysis of the subjective quantifiers.[27] These centrally symmetric Aristotelian diagrams each consist of 4 contradictions, 5 contrarieties, 5 subcontrarieties and 10 subalternations (the latter correspond to the 10 full line arrows in Figure 14). Furthermore, note that there are no Aristotelian relations at all between the L2 bitstrings 1100, 1010, 0011 and 0101 (except for the diagonals of contradiction 1100—0011 and 1010—0101, of course). For Béziau's analysis (Figure 15a), these four bitstrings correspond to the two lexically most primitive and the two lexically most complex subjective quantifier expressions. By contrast, for our own analysis (Figure 15b) these four bitstrings correspond to the quantifier expressions which have intermediate lexical complexity. This is an immediate consequence of the fact that they are bitstrings of level 2, i.e. of intermediate semantic complexity, in combination with the strict correlation between semantic and lexical complexity in our analysis (recall Figure 14).

As the starting point for a third visualisation, note that there exists a logical complementarity between the 6 formulas in the JSB hexagon and the 8 formulas in the Buridan octagon. In terms of Table 3, its upper and lower parts jointly constitute a set of $6 + 8 = 14$ formulas that is Boolean closed.[28] We have recently

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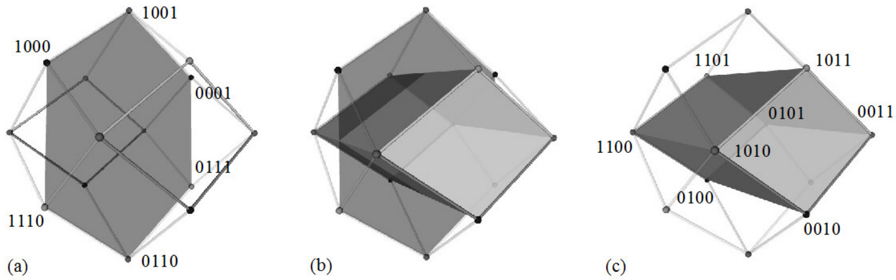
[27] It should be emphasised that we take the term 'Buridan octagon' to refer to (the visualisation of) a certain constellation of Aristotelian relations, rather than to the particular form or content matter of the formulas involved. As a matter of fact, Buridan's own use of his octagon was in describing the logical geometry of first-order modal logic, rather than that of the subjective quantifiers.

[28] I.e. the set is essentially a Boolean algebra with its top and bottom elements left out (recall Footnotes 3 and 4).

FIGURE 16. JSB hexagon (a-b) and Buridan octagon (b-c) embedded in RDH.



shown that this *logical complementarity* between sets of formulas corresponds to a visually appealing *geometrical complementarity* between Aristotelian diagrams [27, 29]. To see this, consider the three-dimensional Aristotelian diagram for the entire Boolean closed set of 14 formulas that was introduced in Section 4, i.e. the rhombic dodecahedron. Figure 16a shows how the JSB hexagon for the 6 'ordinary' quantifiers is embedded into the rhombic dodecahedron. The 8 remaining vertices constitute a 'squeezed' cube, which we have elsewhere called a *rhombicube* [27, 29]. Figure 16c shows how this rhombicube is embedded into the rhombic dodecahedron. Finally, Figure 16b shows the geometrical complementarity between the JSB hexagon of 'ordinary' quantifiers and the rhombicube of subjective quantifiers.

It should be noted that from a strictly logical perspective, there is no difference between the Buridan octagon and the rhombicube: they are merely two different visualisations of a single configuration of 8 formulas and the Aristotelian relations between them. Each of these visualisations has its own advantages and disadvantages. On the one hand, the Buridan octagon is—just like the JSB hexagon—a well-known and canonical diagram, whose visual apprehension is probably facilitated by its two-dimensional nature. On the other hand, the rhombicube stands in a clear visual-geometrical relation of complementarity to the JSB hexagon, and thus better reflects the underlying logical complementarity.

In sum, then, our analysis agrees with that of Béziau's with respect to the JSB hexagon of 'ordinary' quantifiers, but disagrees with respect to the Buridan octagon/rhombicube of subjective quantifiers. In other words, the two analyses share a JSB hexagon, but they complement it with different rhombicubes.

8. Conclusion

In this paper we have evaluated Béziau's Aristotelian diagrams for modalities and quantifiers from a logico-linguistic perspective, and shown how they relate to our framework of logical geometry. In a first main part, we have considered his Jacoby-Sesmat-Blanché stars and stellar rhombic dodecahedron for a set of 12

S5-formulas, and discussed the visualisation of its Boolean closure by means of a rhombic dodecahedron. In a second main part, we have discussed his proposal to transpose his results from S5 to the lexical field of subjective quantification with *many* and *few*, and proposed an alternative analysis based on a number of logical and linguistic considerations. In a final part, we have compared our own analysis with that of Béziau's, making use of a number of two- and three-dimensional Aristotelian diagrams, such as the JSB star/hexagon, the Buridan octagon, and the rhombicube, all of which can be embedded inside the rhombic dodecahedron.

In ongoing work, we are developing a systematic account of the rhombic dodecahedron and its various (families of) subdiagrams [30]. We already have a firm grasp of how the family of JSB hexagons is embedded inside the rhombic dodecahedron; however, in future research we also intend to explore in more detail the embeddings of larger diagrams, such as Béziau's stellar rhombic dodecahedron. Additionally, given the notorious complexity of the subjective quantifiers, it will be interesting to investigate how the two analyses discussed above will hold up under further linguistic scrutiny.

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